Intro to GLM: Binary, Ordered and Multinomial Logistic, and Count Regression Models

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ECPR Summer School in Methods and Techniques

Goals of the course

By the end of this course you should have learned

- How GLM works in general (and how it is implemented)
- How to analyze several common non-linear dependent variables
- How to interpret results of GLMs
- How to present results in a compellign way

Structure of the course

- Monday: Introduction, binary response variables
- Tuesday: How GLM works in general, Maximum Likelihood Estimation
- Wednesday: Results interpretation and quantities of interest
- Thursday: Categorical and ordered response variables
- Friday: Count variables

General considerations

- Usually our theories are about relationships between concepts
- Concepts are measured, so we test relationships between variables
- Modeling is
    1. Describing a relationship between variables
    2. Describing how our concepts are measured, AKA how the data are generated
- GLM takes into account both aspects

Describing relationships between variables

- Suppose we want to study the relationship between education and income: more educated people have higher-paid jobs
- We measure income as the monthly net salary in Euro
- We measure education as the number of years spent in full-time education
- In our model, the total variation of income consists of:
    1. A **systematic** component: how income varies as a function of education
    2. A **stochastic** component: what is due to other causes, which we can not explain with our data
- A model is a summary of the data in terms of the systematic effect + a summary of the magnitude of the unexplained or random variation

Describing relationships between variables (2)

- A **linear** model is an assumption about the nature of the relationship between income an education
- It describes how much income changes *on average* for a unit increase in education
- It also describes how much of the variation of income is *not* explained by education

$$y_i = X_i\beta + e_i$$

- Where the systematic part is the average of Y given a value of X

$$\mu = E(y|X) = X\beta$$

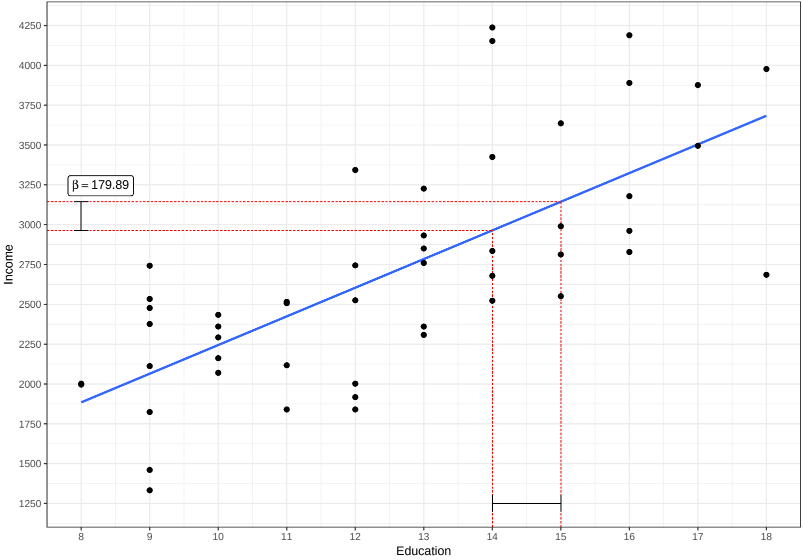
- And the stochastic part is what is left unexplained

$$e_i = y_i - X_i\beta$$

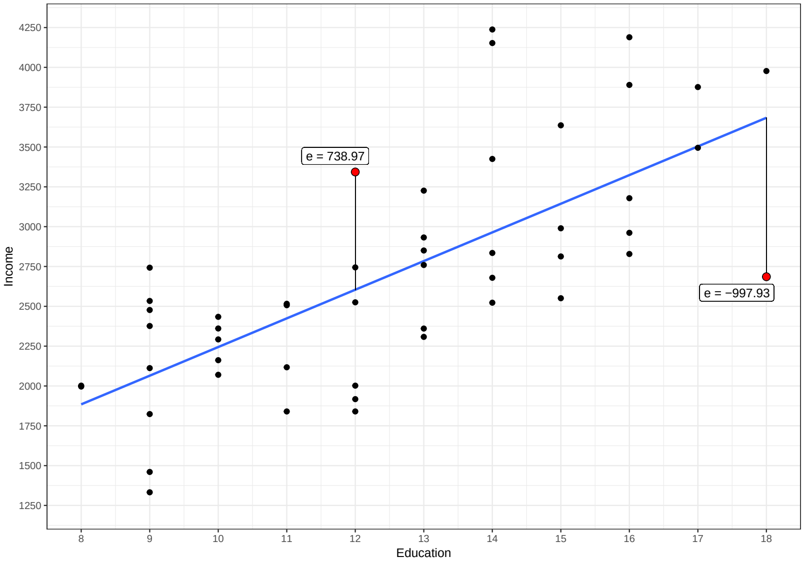
Conceptually

- The systematic component defines the relationship between X and Y, between education and income
  - It looks at the variation of education to explain the variation of income
  - This is what our theories are (usually) about
- The stochastic component defines the distribution of Y
  - It describes the variation of income
  - When we have no predictors (i.e. when we do not know anything about education), *all* the variation of income is stochastic
  - We specify this component by making assumptions about the statistical process that generated the values of income
  - In linear models it is assumed to be "normal"

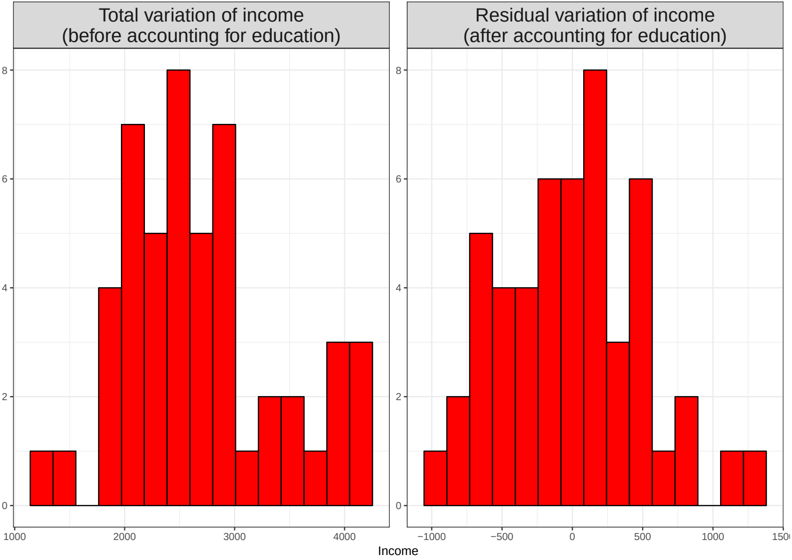
β in practice



e in practice



e in practice (2)



Taking into account how data are generated

- Many social or political event take the form of a yes/no occurrence
  - Did a citizen vote or not?
  - Did a voter choose to vote for the government or for the opposition?
  - Does a person have a job or not?
- What concept do we want to explain here?
- How can we relate other concepts (i.e. independent variables) to it?

The linear probability model

- Sure it is possible to analyze binary responses using linear regression
- This type of model is called **linear probability model**
- Let's consider a voter who has to choose between voting for the *incumbent* party or the *opposition* party

$$y = \begin{cases} 1 & \text{if the incumbent is chosen} \\ 0 & \text{if the incumbent is not chosen} \end{cases}$$

- We can model y as a linear function of people's economic situation compared to the year before
- The more their finances have improved (the higher the value of X) the more likely they will vote for the government

$$y_i = X_i\beta + e_i$$

The linear probability model (2)

- The linear model implies that

$$E(y) = X\beta$$

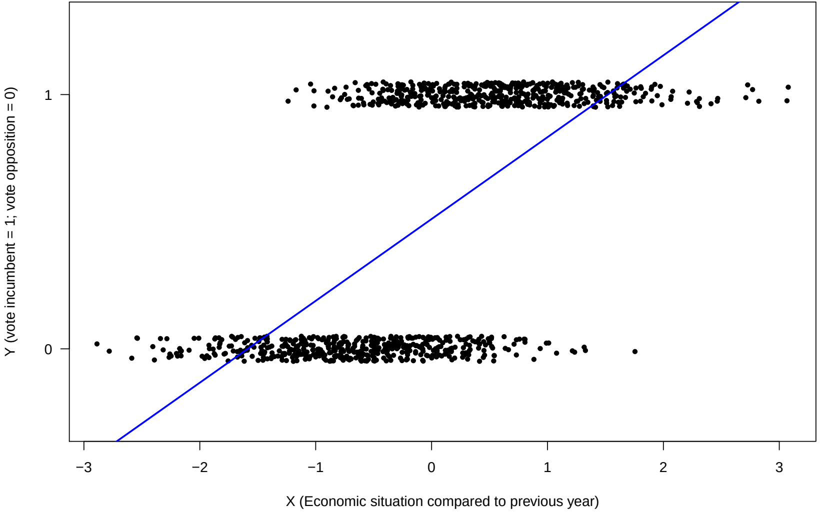
- *E(y)* is the mean of y, which is just the share of y*=1* in our data
- This is interpreted as a probability

$$E(y) = P(y = 1) = \pi$$

- I.e. the linear probability model predicts the mean of y, which is the probability that y has value 1
- It is interpreted in the same way as with linear regression: for 1 point increase in X, β tells how much the probability that y*=1* (that is π) increases

LPM in practice

Example: $Y = 0.51 + 0.32X$



Problems with the LPM

- Besides the violation of normality and homoskedasticity assumptions (which can affect the validity of our results) there are two more immediate concerns:
  1. The LPM makes out-of-bounds predictions
  2. The linear functional form might apply badly to a concept like probability
- The first point is straightforward: what's the predicted value of Y when $X = -2$?
- The second point is trickier
  - The linear functional form implies that π changes at a constant rate, regardless the starting point of the predictor
  - However, this is hardly the case

On probability change

- Example: Bill is choosing whether to buy a product that costs 5€
- One factor influencing the decision is Bill's wealth (X)
- We give him 1€, AKA we increase X of 1 unit
- How much does the probability that Bill buys the product change?

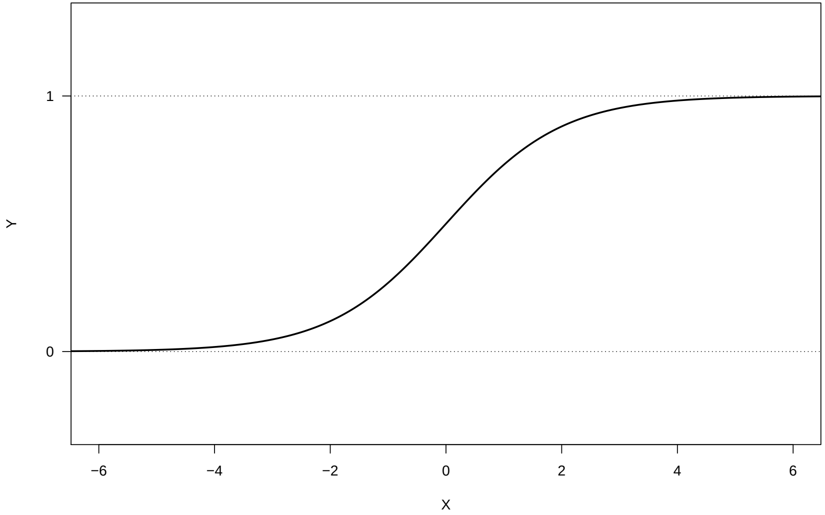
On probability change (2)

- Bill has 0€:
  - Not a great improvement. Bill is still short of 4€, so the probability that he buys the product won't change much
- Bill is millionaire:
  - If he didn't buy the product yet, it's not because of money. Probably he doesn't need it, or he doesn't like it. Again, the change in probability as X increases 1 point will be small
- Bill has 4€
  - Now things are different. By giving Bill 1€, we change his state from not being able to afford the product to being able to do so. Increasing X of 1 unit at this point could have a huge effect

The functional form

- The functional form describes how X relates to Y
- When we model a probability change, we are in fact modeling a discrete event
- This implies that all the possible change of Y can be realized only in one single "step" from 0 to 1
- For this relationship, a **sigmoid** functional form is more appropriate
  - For very low values of X, any increase will have a relatively little impact
  - As we move along the range of X, the effect of one unit increase becomes larger and larger
  - However, passed a certain point, the effect of one unit increase in X becomes smaller again
- To specify the correct functional form is a fundamental step in statistical modeling

Sigmoid relationship



Modeling probabilities with GLM

- The most common ways to model binary outcomes rely on this assumption
- How can we work this out? With GLM
- We need to **transform** the probability of Y (i.e. the mean of Y) in a way such that it can be related to X linerarly
- We do this using a mathematical function called **link function**
- The link function transforms a probability into a quantity called **linear predictor**
- The linear predictor is the systematic component of the model, and can be modeled in the same way as in "simple" linear models

GLM in a nutshell

At the most general level, GLM consists of 3 steps

1. Specify the distribution of the dependent variable
   - This is our assumption about how the data are generated
   - This is the stochastic component of the model
2. Specify the link function
   - We “linearize” the mean of Y by transforming it into the linear predictor
   - It always has an inverse function called **response function**
3. Specify how the linear predictor relates to the independent variables
   - This is done in the same way as with linear regression
   - This is the systematic component of the model

Logit and Probit models

- To model probabilities of binary events, we need a function that maps our linear predictor to a cumulative distribution function
- Two common functions are at the basis of the **logit** and the **probit** models
- The two models work exactly in the same way, except they use a different link function
- Let's consider the linear predictor

$$\eta = X\beta$$

- To be mapped to the probability π with a response function *h()*:

$$\pi = h(\eta) = h(X\beta)$$

Logit models

- We need to find a response function that turns a linear unbounded distribution into a distribution that:
  - Is bounded between 0 and 1
  - Relates to X with a sigmoid functional form
- Logit models use the standard logistic cumulative distribution function:

$$\pi = \frac{exp(\eta)}{1 + exp(\eta)} = \frac{exp(X\beta)}{1 + exp(X\beta)}$$

- And the link function is called **logit function**:

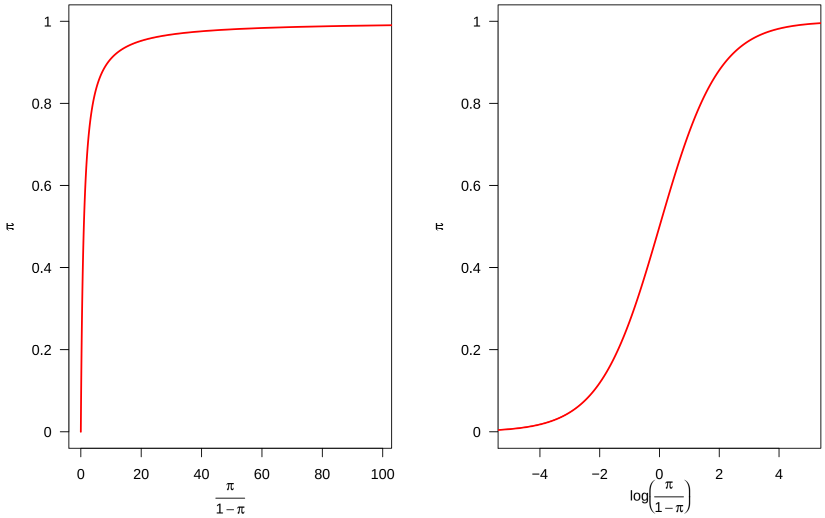
$$\eta = X\beta = log\left(\frac{\pi}{1 - \pi}\right)$$

- The part $\left(\frac{\pi}{1-\pi}\right)$ is called "odds", and refers to the probability to observe an event versus its complement

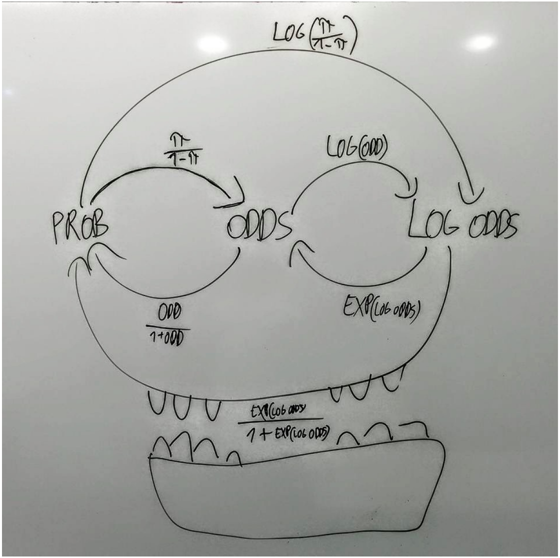
Probabilities, odds, and log odds

Probability π	Odds $\frac{\pi}{1-\pi}$	Logits $log\left(\frac{\pi}{1-\pi}\right)$
0.01	$1/99 = 0.0101$	-4.60
0.05	$5/95 = 0.0526$	-2.94
0.10	$1/9 = 0.1111$	-2.20
0.30	$3/7 = 0.4286$	-0.85
0.50	$5/5 = 1$	0.00
0.70	$7/3 = 2.3333$	0.85
0.90	$9/1 = 9$	2.20
0.95	$95/5 = 19$	2.94
0.99	$99/1 = 99$	4.60

Probabilities, odds, and log odds (2)



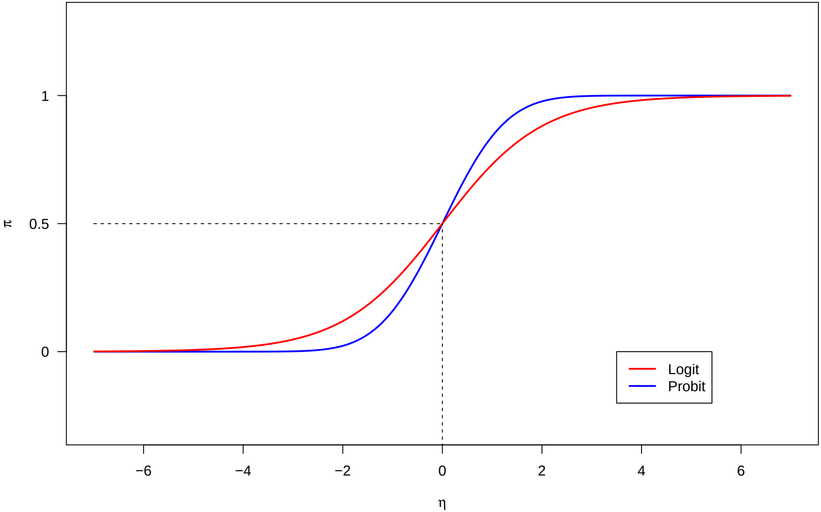
Probabilities, odds, and log odds (3)



Probit models

- In probit models, the response function *h()* is the standard normal CDF:
$$\pi = \Phi(\eta) = \Phi(X\beta)$$
- And the link function *g()* is the inverse:
$$\pi = \Phi^{-1}(\eta) = \Phi^{-1}(X\beta)$$
- However, the inverse function Φ^{-1} has no easy analytic solution, so it is found numerically

Logit vs. Probit functions



Logit and Probit models

- Note from the figure that both functions are nearly linear for the most of their range
  - In fact the linear probability model leads to similar results, except for extreme values of Y
- Logit and probit models produce identical predicted values, but different coefficients
- Models using the logit link function are more common than probit models
- This is also a matter of ease of interpretation:
- Essentially, logit models are **linear models for log-odds**

A latent variable interpretation

- Binary response variables can be regarded more directly as a measurement problem
- We can think of a continuous unobservable construct $y*$, e.g. the propensity to turnout at the next election
- We can't observe $y*$, we can only observe its manifest variable y in two states, e.g. whether a persone says s/he will vote at the next election or not
- In fact, a voter might be barely convinced to turn out, while another might be enthusiastic about the election
- However, all we see is the discrete choice whether they will vote (1) or not (0)

A latent variable interpretation (2)

- $y*$ is linked to y by the measurement equation:

$$y_i = \begin{cases} 0 & \text{when } y*_i \leq 0 \\ 1 & \text{when } y*_i > 0 \end{cases}$$

- The value *0* is an arbitrary threshold on $y*$: when it is passed, y switches from *0* to *1*
- In this context we model:

$$y*_i = X_i\beta + e_i$$

- And the probability that $y_i = 1$ is:

$$P(y*_i > 0) = P(X_i\beta + e_i > 0)$$

A latent variable interpretation (3)

- Since $y*$ is not observed, we can't estimate its variance: we need to fix it at a given value
- Different assumptions about the variance of e lead to different model specifications:
  - If $Var(y*) = \pi^2/3$, $y*$ follows a standard logistic distribution
  - If $Var(y*) = 1$, $y*$ follows a standard normal distribution
- Depending on which distribution of e we assume, solving the equation in the previous slide produces formulations that are equivalent to the logit or the probit model
- This approach requires more theorization – i.e. we need to find a convincing definition of the latent variable
- However, in practice it produces identical results

In sum

- Binary responses can not be related to our predictors linearly
- To model them, we need to transform their distribution in a way that can be treated as in a linear model
- GLM requires us to:
  - Make an assumption about the distribution of y
  - Find a link function to make the distribution of y linear
  - Model the transformed linear predictor

What we will see tomorrow

- How GLM for binary responses works with individual data
- How parameters in GLM are estimated