Intro to GLM – Day 4: Multiple Choices and Ordered Outcomes

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Categorical events with more than two outcomes

In social science, many phenomena do not consist of simple yes/no alternatives

1. **Categorical** variables
   - Example: multiple choices
   - A voter in a multiparty system can choose between many political parties
   - A consumer in a supermarket can choose between several brands of toothpaste
2. **Ordinal** variables
   - Survey questions often ask "how much do you agree" with a certain statement
   - You may have 2 options: "agree" or "disagree"
   - You may have more options: e.g. "completely agree", "somewhat agree", "somewhat disagree", "completely disagree"

Categorical dependent variables

- Imagine a country where voters can choose between 3 parties: "A", "B", "C"
- We want to study whether a set of individual attributes affect vote choice
- In theory, we could run several binary logistic regressions predicting the probability to choose between any two parties
- If we have three categories, how many binary regressions do we need to run?

Multiple binary models?

- We need to run only 2 regressions:

$$log\left[\frac{P(A|X)}{P(B|X)}\right] = \beta_{A|B}X; \quad log\left[\frac{P(B|X)}{P(C|X)}\right] = \beta_{B|C}X$$

- Estimating also $log\left[\frac{P(A|X)}{P(C|X)}\right]$ would be redundant:

$$log\left[\frac{P(A|X)}{P(B|X)}\right] + log\left[\frac{P(B|X)}{P(C|X)}\right] = log\left[\frac{P(A|X)}{P(C|X)}\right]$$

- And:

$$\beta_{A|B}X + \beta_{B|C}X = \beta_{A|C}X$$

Multiple binary models? (2)

- However, if we estimated all binary models independently, we would find out that $\beta_{A|B}X + \beta_{B|C}X \neq \beta_{A|C}X$
- Why? Because **the samples would be different**
- The model for $log\left[\frac{P(A|X)}{P(B|X)}\right]$ would would include only people who voted for "A" *or* "B"
- The model for $log\left[\frac{P(B|X)}{P(C|X)}\right]$ would would include only people who voted for "B" *or* "C"
- We want a model that uses the full sample and estimates the two groups of coefficients simultaneously

Multinomial probability model

- To make sure that the probabilities sum up to *1*, we need to take all alternatives into account in the same probability model
- As a result, the probability that a voter *i* picks a party *m* among a set of *J* parties is:

$$P(Y_i = m|X_i) = \frac{exp(X_i\beta_m)}{\sum_{j=1}^{J} exp(X_i\beta_j)}$$

- **Note**: to make sure the model is identified, we need to set $\beta = 0$ for a given category, called the "baseline category"
- Conceptually, this is the same as running only 2 binary logit models when there are 3 categories

Multinomial probability model (2)

- We can still obtain predicted probabilities for each category
- Assuming that the baseline category is *1*, the probability of $Y = 1$ is:

$$P(Y_i = 1|X_i) = \frac{1}{1 + \sum_{j=2}^{J} exp(X_i\beta_j)}$$

- And the probability of $Y = m$, where m refers to any other category, is:

$$P(Y_i = m|X_i) = \frac{exp(X_i\beta_m)}{1 + \sum_{j=2}^{J} exp(X_i\beta_j)} \text{ for } m > 1$$

- The choice of the baseline category is arbitrary
- However, it makes sense to pick a theoretically meaningful one

Estimation of multinomial logit models

- The likelihood function for the multinomial logit model is:

$$L(\beta_2, \ldots, \beta_j | y, X) = \prod_{m=1}^{J} \prod_{y_j = m} \frac{exp(X_i \beta_m)}{\sum_{j=1}^{J} exp(X_i \beta_j)}$$

- Where $\prod_{y_j=m}$ is the product over the cases where $y_i = m$
- The estimation will work as usual: the software will take the log-likelihood function and it will look for the ML estimates of β iteratively
- For every independent variable, the model will produce $J - 1$ parameter estimates

Multinomial logit: interpretation

- Like in binary logit, our coefficients are log-odds to choose category m instead of the baseline category

$$exp(X_i\beta_m) = \frac{\pi_m}{\pi_1}$$

- How do we compare the coefficients between categories that are not the baseline?
- First, again, pick a baseline category that makes sense
- Second, comparing coefficients between estimated categories is straightforward:

$$\frac{\pi_m}{\pi_j} = exp[X_i(\beta_m - \beta_j)]$$

- I.e. the exponentiated difference between the coefficients of two estimated categories is equivalent to the odds to end up in one category instead of the other (given a set of individual characteristics)

Multinomial logit: predicted probabilities

- Predicted probabilities to choose any of the estimated categories are:

$$\pi_{im} = \frac{exp(X_i\beta_m)}{1 + \sum_{j=2}^{J} exp(X_i\beta_j)}$$

- And for the baseline category they are:

$$\pi_{i1} = \frac{1}{1 + \sum_{j=2}^{J} exp(X_i\beta_j)}$$

Multinomial models as choice models

- A way to interpret multinomial models is, more directly, as *choice* models
- This approach is sometimes called “Random Utility Model” and it is quite popular in economics
- This interpretatons is based on two assumptions:
  - *Utility* varies across individuals. Different individuals have different utilities for different options
  - Individual decision makers are *utility maximizers*: they will choose the alternative that yields the highest utility
- Utility: the degree of satisfaction that a person expects from choosing a certain option
- The utility is made of a systematic component μ and a stochastic component e

Utility and multiple choice

- For an individual i, the (random) utility for the option m is:

$$U_{im} = \mu_{im} + e_{im} = X\beta_{im} + e_{im}$$

- When there are J options, m is chosen over an alternative $j \neq m$ if $U_{im} > U_{ij}$

$$P(Y_i = m) = P(U_{im} > U_{ij})$$
$$P(Y_i = m) = P(\mu_{im} - \mu_{ij} > e_{ij} - e_{im})$$

- The likelihood function and estimation are identical to the probability model that we just saw

Assumptions

1. The stochastic component follows a Gumbel distribution (AKA "Type I extreme-value distribution")

$$F(e) = exp[-e - exp(-e)]$$

2. Among different alternatives, the errors are identically distributed
3. Among different alternatives, the errors are independent
   - This assumptions is called "independence of the irrelevant alternatives", and it is quite controversial
   - It states that the ratio of choice probabilities for two different alternatives is independent from all the other alternatives
   - In other words, if you are choosing between party "A" and party "B", the presence of party "C" is irrelevant

Conditional logit

- In multinomial logit models, we explain choice beween different alternatives using attributes of the decision-maker
- E.g. education, gender, employment status
- However, it is possible to explain choice using attributes of the alternatives themselves
- E.g. are voters more likely to vote for bigger parties?
- The latter model is called "conditional logit"
- It is not so common in political science, as it requires observing variables that vary between the choice options

Multinomial vs Conditional logit

Multinomial logit

- We keep the values of the predictors constant across alternatives
- We let the parameters vary across alternatives
  - E.g. the gender of a voter is always the same, no matter if s/he's evaluating party "A" or party "B"
  - The effect of gender will be different between party "A" and "B"

Conditional logit

- We let the values of the predictors change across alternatives
- We keep the parameters constant across alternatives
  - The size of party "A" and party "B" is the same for all individuals
  - The effect of size is the same for all parties

Ordinal dependent variables

- Suppose the categories have a natural order
- For instance, look at this item in the World Values Study:
- "*Using violence to pursue political goals is never justified*"
  - Strongly Disagree
  - Disagree
  - Agree
  - Strongly Agree
- Here we can rank the values, but we don't know the distance between them
- We could use a multinomial model, but this way we would ignore the order, losing information

Modeling ordinal outcomes

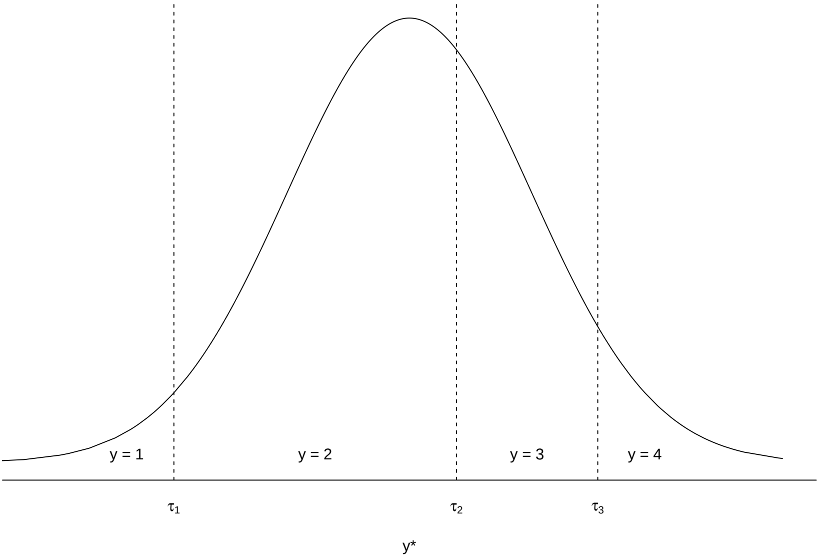
- Two ways of modeling ordered categorical variables:
  - A latent variable model
  - A non-linear probability model
- These two methods reflect what we have seen with binary response models
- In fact, you can think of binary models as special cases of ordered models with only 2 categories
- As with binary models, the estimation will be the same
- However, for ordered models, the latent variable specification is somewhat more common

A latent variable model

- Imagine we have an unobservable latent variable $y*$ that expresses our construct of interest (e.g. endorsement of political violence)
- However, all we can observe is the ordinal variable y with M categories
- $y*$ is mapped into y through a set of cut points τ_m

$$y_i = \begin{cases} 1 & \text{if } -\infty < y_i* < \tau_1 \\ 2 & \text{if } \tau_1 < y_i* < \tau_2 \\ 3 & \text{if } \tau_2 < y_i* < \tau_3 \\ 4 & \text{if } \tau_3 < y_i* < +\infty \end{cases}$$

Cut points



A latent variable model (2)

- Like with the binary model, $y*$ is a function of both a systematic and a stochastic component

$$y_i* = X_i\beta + e_i$$

- Then, the model is essentially a linear regression of $y*$
- To be able to estimate the model we need to:
  - Fix the variance of e to an assumed value
    - Either *1* (then e is normally distributed)
    - Or $\pi^2/3$ (then e il logistically distributed)
  - Exclude the constant term from the estimation of the parameters
    - Instead, estimated values of τ_1, τ_2, ..., τ_{M-1} serve as intercepts
    - Where M is the number of categories

A non-linear probability model

- Ordinal models can be also seen as models of the **cumulative probability** that an outcome y is less than or equal to m
- So, instead of modeling the probability that a certain event happens (like in binary models), here we model the probability of an event *and of all events that are ordered before it*:

$$P(y_i \leq m|X_i) = \sum_{j=1}^{m} P(y_i = j|X_i)$$

- In terms of odds, it is the odds that $y \leq m$ vs $y > m$:

$$\Omega_{im}(X_i) = \frac{P(y_i \leq m|X_i)}{1 - P(y_i \leq m|X_i)} = \frac{P(y_i \leq m|X_i)}{P(y_i > m|X_i)}$$

Probability model

- The cumulative probability to observe an outcome of $y \leq m$ is:

$$P(y_i \leq m|X_i) = F(\tau_m - X_i\beta)$$

- And the probability to observe an outcome of $y = m$ Is:

$$P(y_i = m|X_i) = F(\tau_m - X_i\beta) - F(\tau_{m-1} - X_i\beta)$$

- Where *F()* is either the standard normal or logistic CDF
- Again, the choice of the link function determines whether we estimate an *ordered logit* or an *ordered probit* model

Estimation of ordered models

- The likelihood function for ordered models is:

$$L(\beta, \tau | y, X) = \prod_{j=1}^{J} \prod_{y_i = m} \left[F(\tau_m - X_i\beta) - F(\tau_{m-1} - X_i\beta) \right]$$

- Where $\prod_{y_i=m}$ indicates to multiply over the cases where $y = m$
- As usual, the software will plug in the link function, take the log-likelihood function and look for the ML estimates of β and τ

Proportional odds assumption

- In the probability function that we have seen, β is the same regardless which categories we are considering, while τ is different
- This is equivalent to estimate a number of parallel regression lines, where only the intercept changes
- For instance, if y has 4 categories:

$$P(y_i \leq 1|X_i) = F(\tau_1 - X_i\beta)$$

$$P(y_i \leq 2|X_i) = F(\tau_2 - X_i\beta)$$

$$P(y_i \leq 3|X_i) = F(\tau_3 - X_i\beta)$$

- In logit models this is called the "proportional odds assumption"
- It can be tested comparing the β obtained by an ordered regression with a set of βs obtained by a set of binary regressions for each $P(y_i \leq m|X_i)$

Ordered logit: interpretation

- Unlike the multinomial logistic model, we have only one set of βs here
- This is due to the "proportional odds" assumption, which implies that our βs are the same for each cut point τ_m
- As we are accustomed to think, the coefficients are log-odds to choose category m instead of a lower category

$$exp(X_i\beta_m) = \frac{\pi_m}{\pi_{m-1}}$$

- Also the values of τ are on the same scale: they indicate the log-odds to be in a category below the cut point when all predictors are equal to zero

Ordered logit: interpretation (2)

- In ordered models, we can predict two types of probabilities:
  - The *cumulative* probability, i.e. the probability that y will be in the category m or in a lower ranked category
  - The probability that y is in a specific category
- If we use the standard logistic CDF as link function, the formula to get cumulative predicted probabilities is:

$$P(y_i \leq m|X_i) = \frac{exp(\tau_m - X_i\beta)}{1 + exp(\tau_m - X_i\beta)}$$

Ordered logit: interpretation (3)

- To get predicted probabilities for specific categories, we must still take the cumulative probability and subtract the predicted probability for the lower ranked category:

$$P(y_i = m) = \frac{exp(\tau_m - X_i\beta)}{1 + exp(\tau_m - X_i\beta)} - \frac{exp(\tau_{m-1} - X_i\beta)}{1 + exp(\tau_{m-1} - X_i\beta)}$$

- Note that the larger the difference between τ_m and τ_{m-1}, the easier it will be to answer $y_i = m$.
- This is the case in some survey items where many people choose the middle category